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COMBINATORICS OF SIMPLE CLOSED CURVES ON THE TWICE PUNCTURED TORUS

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To Joan Birman on her 70th birthday

ABSTRACT

In the standard enumeration of homotopy classes of curves on a surface as words in a generating set for the fundamental group it is a very hard problem to discern those that are simple. In this paper we describe how the complex of simple closed curves on a twice punctured torus Σ may be given a strikingly simple description by representing them as homotopy classes of **paths** in a **groupoid** with two base points. Our starting point are the π_1-train tracks developed by Birman and Series. These are weighted train tracks parameterizing the simple closed curves on Σ similar to Thurston's, but they are defined relative to a fixed presentation of

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$\pi_1(\Sigma)$. We approach the problem by cutting the surface into two disjoint "cylinders"; this decomposes the π_1-train tracks into two disjoint parts, relative to which all patterns and relations become much more transparent, each part reducing essentially to the well-known case of a once punctured torus. We obtain global coordinates, called $\pi_{1,2}$-weights, for simple closed loops. These coordinates can be easily identified with Thurston's projective measured lamination space S^3. We also solve the problem which originally motivated this work by proving a simple relationship between the leading terms of traces of simple loops in a holomorphic family of representations ρ: $\pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbf{C})$ (corresponding to the Maskit embedding of the twice punctured torus) and the $\pi_{1,2}$-weights.

1. Introduction

The main objects of study in this paper are the simple closed curves on a twice punctured torus Σ. We introduce a new model whereby every free homotopy class of simple loops γ on Σ is assigned to a point $\mathbf{i}(\gamma) \in (\mathbf{Z}^+ \times \mathbf{Z})^2$ called its $\pi_{1,2}$-**coordinates**. Using simple combinatorial procedures, the coordinates $\mathbf{i}(\gamma)$ both determine, and are determined by, standard representations of γ as a weighted train track or as a word in $\pi_1(\Sigma)$. In addition, however, the $\pi_{1,2}$-coordinates reveal significant new information.

Theorem 5.3 reveals a deep connection between the $\pi_{1,2}$-coordinates of a simple loop γ on Σ and the expression of γ as a word in a given set of generators for $\pi_1(\Sigma)$: it addresses, in a precise way, the problem of recognizing patterns in the words to say whether or not the loop is simple.

We apply Theorem 5.3 to a problem that arose in studying the Maskit embedding of the Teichmüller space of Σ as a holomorphic family of Kleinian groups parametrized by a subset of $\mathbf{C}^2$. The parameters for this family are chosen so they are also the plumbing parameters of Kra [8]. The traces of the group elements are polynomials in the parameters. Let $g \in \pi_1(\Sigma)$ represent a simple closed curve γ on Σ. We prove a remarkable formula, Theorem 6.1, that gives the order and coefficients of the top three terms of the trace of g explicitly in terms of $\mathbf{i}(\gamma)$. This formula generalizes the analogous formula for once punctured tori given in [6]. Similar formulae for four and five time punctured spheres are proved using different methods in [4].

In [7], we will use Theorem 6.1 in connection with the theory of pleating coordinates (introduced in [6]) to deduce detailed and precise information about the shape of the Maskit embedding in $\mathbf{C}^2$.

In the final section, we use $\pi_{1,2}$-coordinates to make explicit the S^3 structure of Thurston's projective measured lamination space for Σ.

Our model is motivated by the π_1-train tracks introduced by Birman and Series in [1]. These are train tracks in the usual sense, but carry additional group theoretic information. A π_1-train track is defined relative to a fixed fundamental domain R for the action of $\pi_1(\Sigma)$ acting on the hyperbolic plane. It is a train track, in the sense of Thurston, all of whose switches lie on ∂R with at most one switch on each side. The switches are named by the corresponding side pairing transformations of R, which are a set of geometric generators of $\pi_1(\Sigma)$. The point is that not only do multiple simple loops on Σ correspond to weighted train tracks in the usual way, but that, in addition, there is an immediate relationship between the weighted π_1-train track associated to a simple loop γ and the word $w(\gamma)$ in the geometric generators representing γ in $\pi_1(\Sigma)$. More precisely, $w(\gamma)$ is exactly the sequence of switches traversed in order by γ, and conversely, the weighted π_1-train track may be read off mechanically from $w(\gamma)$.

In this paper, we begin with a specific presentation for the fundamental group $\pi_1(\Sigma, p)$, $p \in \Sigma$, but show that for many purposes it is simpler and easier to decompose Σ into two cylindrical subsurfaces by cutting along a pair of disjoint curves, one passing through each puncture. The point of this decomposition is that we can simplify the π_1-train track by looking at its restriction to each cylinder. We call these restricted train tracks **$\pi_{1,2}$-train tracks** because they are related to the fundamental groupoid $\pi_{1,2}(\Sigma, p_1, p_2)$, that is, the groupoid of homotopy classes of paths in Σ with endpoints in the set $\{p_1, p_2\}$, where one basepoint is chosen on each cylinder.

Once again, a simple closed loop γ on Σ defines a unique weighted $\pi_{1,2}$-train track. Our $\pi_{1,2}$-coordinates $\mathbf{i}(\gamma)$ are easy functions of these weights. Morally, the components of $\mathbf{i}(\gamma)$ are the intersection numbers of the simple loops with the longitudinal and meridional curves on the cylinders. Conversely, from points in $(\mathbf{Z}^+)^2 \times \mathbf{Z}^2$, we recover weighted $\pi_{1,2}$-train tracks and read off directly corresponding cyclically reduced words both in the groupoid and the group. (See Theorem 4.5.) The patterns for simple loops on Σ in terms of these groupoid words are strikingly easy to recognize and this is the basis of Theorem 5.3.

Our result on trace polynomials is obtained by factoring a representation of $\pi_1(\Sigma)$ in $\mathrm{SL}(2, \mathbf{C})$ as a representation of the groupoid $\pi_{1,2}(\Sigma)$. Effectively, this involves factoring the image of one of the generating loops into a product of two matrices representing two open paths in the groupoid whose product is the loop. The factored product has a surprisingly simple form from which we are able to obtain our result.

We believe that the techniques introduced in this paper will have significant extensions to surfaces of higher genus and intend to explore these possibilities elsewhere.

This paper is arranged as follows: In section 2 we set notation and discuss cutting sequences and the fundamental groupoid. In section 3 we introduce $\pi_{1,2}$-train tracks and in section 4 define $\pi_{1,2}$-coordinates. In section 5 we prove Theorem 5.3. The application to trace polynomials is given in section 6. Finally, in section 7, we use $\pi_{1,2}$-coordinates to give an explicit embedding of projective measured lamination space as S^3. Throughout the paper $\mathbf{Z}^+$ denotes $\{0, 1, 2, 3, \ldots\}$.

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2. Fundamental concepts

2.1 THE TWICE PUNCTURED TORUS. Let Σ be the twice punctured torus with fundamental group $G = \pi_1(\Sigma, p)$, the homotopy classes of loops on Σ with basepoint $p \in \Sigma$. Choosing some hyperbolic metric on Σ, we may realize the universal covering space of Σ as the unit disk D with the hyperbolic metric. Let R be a fundamental domain for the action of G on D. We choose R to be a six sided geodesic polygon with vertices on ∂D that project to the punctures of Σ as shown in Figure 1.

Label the vertices of R in clockwise order around D by $v_1, \ldots, v_6$. We name the side pairing identifications of R as follows (see Figure 1):

S_1 identifies v_1v_2 with v_4v_3
S_2 identifies v_1v_6 with v_4v_5
T identifies v_6v_5 with v_2v_3

These pairings determine a presentation of G as the free group on S_1, S_2, T. We write G_0 for the generating set $\{S_1^{\pm 1}, S_2^{\pm 1}, T^{\pm 1}\}$ and, when convenient, we write $\bar{X}$ for X^{-1}, $X \in G_0$.

In Figure 1 we draw R with its basepoint $\mathbf{p}$ (a lift of p), and some of the adjacent regions. The arc in Figure 1 from $\mathbf{p}$ to $S_1\mathbf{p}$ projects to a curve on Σ in the homotopy class in $\pi_1(\Sigma, p)$ corresponding to the element S_1. We label the common side of R and S_1R by $\bar{S}_1$ on the side inside R and by S_1 on the side inside S_1R. We label the remaining sides analogously. Thus the side X is mapped to the side $\bar{X}$ by the generator X for each $X \in G_0$.

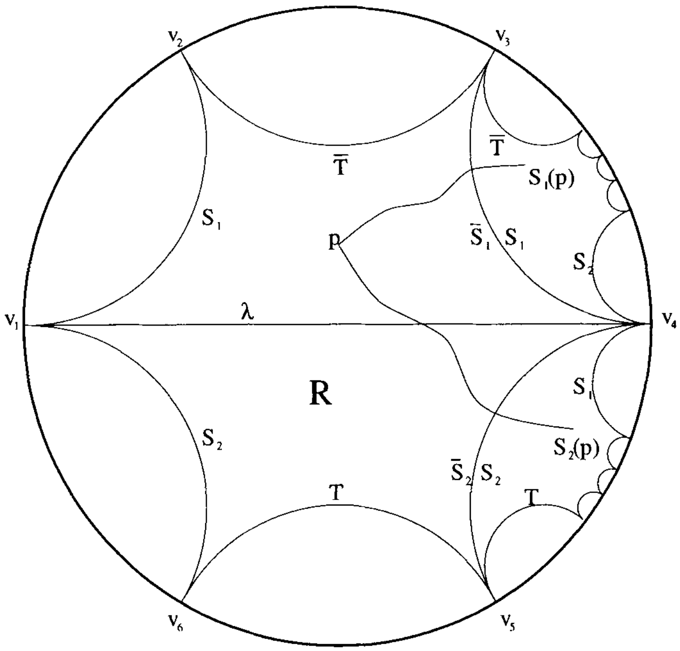


Figure 1. The fundamental domain R.

Starting from the side v_1v_2, the 6 sides interior to R are labelled in clockwise order $\{S_1, \bar{T}, \bar{S_1}, \bar{S_2}, T, S_2\}$. If we need to distinguish between the side and the transformation we call the side $s(X)$ (note that $s(X)$ does not include the vertices). We transport the side labelling to the full tiling of D by G and call this the **G-labelling**.

2.2 G-CUTTING SEQUENCES. We recall the method of cutting sequences for representing loops on a hyperbolic surface developed in, for example, [2, 11].

Let γ be any closed curve on Σ. Let $\tilde{\gamma}$ be a lift that starts on a side of an image of R and projects bijectively to γ. Then $\tilde{\gamma}$ intersects in order the interiors of the images $g_0R, g_1R, \ldots, g_{k-1}R$ of R ending on the common side of $g_{k-1}R$ and g_kR. Thus $\tilde{\gamma}$ is divided into k segments whose translates back to R, $\tilde{\gamma}_i = g_i^{-1}\tilde{\gamma} \cap R$, $i = 0, \ldots, k-1$ we call the **G-segments of** γ. Let X_i be the label of the side between $g_{i-1}R$ and g_iR, interior to g_iR, $i = 1, \ldots k$. We call $X_1 \ldots X_k$ the **G-cutting sequence** of γ. Note that if g represents γ in G, so that $g_k = g_0g$, then

$$g = g_0^{-1}g_1 \cdot g_1^{-1}g_2 \cdots g_{k-1}^{-1}g_k = X_1X_2 \cdots X_k.$$

The G-segments for the loop γ^* with cutting sequence $T\bar{S}_2\bar{S}_1T\bar{S}_2S_1\bar{T}S_1$ are drawn in Figure 2.

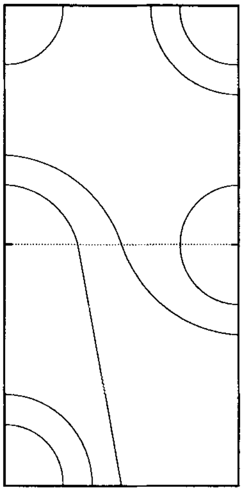


Figure 2. The cutting sequence of γ^* is $T\bar{S}_2\bar{S}_1T\bar{S}_2S_1\bar{T}S_1$.

2.3 REDUCED AND CYCLICALLY REDUCED SEQUENCES. We say a G-cutting sequence $X_1 \cdots X_k$, $X_r \in G_0$ is **reduced** if $X_{r-1} \neq \bar{X}_r$, $r = 2, \ldots, k$. It is **cyclically reduced** if in addition $X_1 \neq \bar{X}_k$. A loop γ is **reduced** if its cutting sequence is cyclically reduced. We have (see [1, 2]):

THEOREM 2.1: *There are bijective correspondences between*

- *Free homotopy classes of reduced loops on* Σ
- *Cyclically reduced words* $W = X_1X_2\cdots X_k$, $X_i \in G_0$ *modulo cyclic permutation*
- *Conjugacy classes in* G.

2.4 THE FUNDAMENTAL GROUPOID. With slight modifications, the above picture can be used to define cutting sequences associated with the fundamental groupoid on Σ with two base points, $\pi_{1,2}(\Sigma, p_1, p_2)$.

This idea is of central importance in what follows.

Definition: A groupoid $\mathcal{G}$ is a pair $(X, \mathcal{G})$ where X is a base space and the elements of $\mathcal{G}$ are arrows with endpoints in X. There are two maps $l, r\colon \mathcal{G} \to X$ that map $\gamma \in \mathcal{G}$ to its initial and final points, respectively. The product $\gamma_1\gamma_2$ is defined if and only if $r(\gamma_1) = l(\gamma_2)$ and then $l(\gamma_1\gamma_2) = l(\gamma_1)$, $r(\gamma_1\gamma_2) = r(\gamma_2)$. All the arrows have inverses; furthermore, $l(\gamma^{-1}) = r(\gamma)$ and vice versa.

Definition: The **fundamental 2-point groupoid** of Σ, $\pi_{1,2}(\Sigma, p_1, p_2)$ is the groupoid with base space $B = \{p_1, p_2\}$, $p_i \in \Sigma$, $p_1 \neq p_2$, and arrows consisting of homotopy classes of paths in Σ with initial and final points in B.

We realize $\Gamma = \pi_{1,2}(\Sigma, p_1, p_2)$ as part of the setup of section 2.2 as follows. In the region R, join the vertices v_1 and v_4 by a line λ as in figure 1. This divides R into 2 quadrilaterals R_1 and R_2 with vertices $\{v_1, v_2, v_3, v_4\}$ and $\{v_1, v_4, v_5, v_6\}$, respectively. Assume, without loss of generality, that $\mathbf{p} \in R_1$ and set $\mathbf{p_1} = \mathbf{p}$. Choose a point $\mathbf{p}' \in R_2$ and set $\mathbf{p_2} = \mathbf{p}'$. We take the projections of $\mathbf{p_1}, \mathbf{p_2}$ as the base space $\{p_1, p_2\}$ for the groupoid Γ. Let $\tilde{s}_1$ be a path joining $\mathbf{p_1}$ and $S_1\mathbf{p_1}$; $\tilde{s}_2$ a path joining $\mathbf{p_2}$ and $S_2\mathbf{p_2}$; $\tilde{t}_2$ a path in R from $\mathbf{p_2} \in R_2$ to $\mathbf{p_1}$ in R_1 crossing λ; and $\tilde{t}_1$ a path from $\mathbf{p_1}$ to $\mathbf{p_2}$, leaving R_1 across the side v_2v_3 and reentering R into R_2 across the side v_6v_5. We denote the projections of the homotopy classes of these paths to Σ by s_1, s_2, t_1, t_2 and the same paths in the opposite directions by $s_1^{-1}, s_2^{-1}, t_1^{-1}, t_2^{-1}$. We write $\Gamma_0 = \{s_i^{\pm 1}, t_i^{\pm 1}\}$. Where convenient we write $\bar{y}$ for y^{-1}, $y \in \Gamma_0$.

2.5 Γ-CUTTING SEQUENCES. We associate cutting sequences to Γ as follows. We label the (oriented) sides of the regions R_i defined above by symbols $\{s_1, \bar{s}_1, s_2, \bar{s}_2, t_1, \bar{t}_1, t_2, \bar{t}_2\}$ in such a way that the path called $y \in \Gamma_0$ above has cutting sequence y. In other words, the labels $s_i, \bar{s}_i$ are given to the sides of R that have the G-labels $S_i, \bar{S}_i$ respectively. The label t_1 is given to the side v_5v_6 of R_2 inside R_2, and the label t_2 is given to the side v_1v_4 inside R_1. The other side of v_1v_4, inside R_2, gets the label $\bar{t}_2$ and the side v_2v_3 inside R_1 gets the label $\bar{t}_1$. Again, where convenient, we refer to sides as $s(y), y \in \Gamma_0$.

We refer to this as the **Γ-labelling**. The Γ-labelling, like the G-labelling above, is transported to all of the images of R_1 and R_2 under the action of G.

Just as we did for the G-labelling, we can use the Γ-labelling to form Γ-cutting sequences of any path α on Σ with endpoints in $\{p_1, p_2\}$. We get sequences, $y_1 \cdots y_l$, $y_i \in \Gamma_0$.

A lift of a loop γ on Σ to D is decomposed into segments by its intersections with sides of R_1 and R_2. Projecting back down to Σ, we obtain elements of Γ. Lifting these paths to R_1 and R_2, we obtain a family of segments that we call the **Γ-segments** of γ (in analogy with the G-segments defined in section 2.2).

Using Γ-cutting sequences it is easy to see that the paths in Γ_0 are a set of free generators. We have a discussion about reduced and cyclically reduced sequences completely analogous to that in section 2.3.

2.6 RELATIONS BETWEEN CUTTING SEQUENCES. In what follows, we will need to convert from Γ to G-cutting sequences.

Every element of G is a product of the generating loops T, S_1, S_2 and their inverses. Since each loop based at p_1 is also an element of Γ, we find by inspection the expressions for T, S_1, S_2 in terms of elements of Γ:

$$(**) \qquad t_1 t_2 = T, \quad \bar{t}_2 s_2 t_2 = S_2, \quad s_1 = S_1.$$

These relations may be interpreted either on the level of equivalent cutting sequences, or as equalities between homotopy classes of paths (loops) on Σ. Thus any word in G expressed in terms of the generators G_0 may be expressed using relations (**) as a word in the generators Γ_0.

3. Simple loops and train tracks

3.1 MULTIPLE SIMPLE LOOPS. In this section we review briefly the relevant facts about π_1-train tracks as introduced in [1] and then extend this notion to $\pi_{1,2}$-train tracks associated to the fundamental groupoid $\Gamma = \pi_{1,2}(\Sigma)$. First we recall some basic definitions.

A loop γ on Σ is **boundary parallel** if it is homotopic to a loop around a puncture.

A simple loop on Σ is a loop with no self-intersections. A **multiple simple loop** $\gamma = \{\gamma_1, \ldots, \gamma_k\}$ is a collection of pairwise disjoint simple loops on Σ, none of which is boundary parallel.

The maximal number of distinct pairwise disjoint non-boundary parallel homotopy classes of simple loops on Σ is two. Thus any multiple simple loop γ on Σ consists of m_i disjoint copies of loops γ_i, $i = 1, 2$, where γ_1 and γ_2 are homotopically distinct and not boundary parallel and $m_i \in \mathbf{Z}^+$. For convenience we write $\gamma = m_1\gamma_1 + m_2\gamma_2$.

3.2 π_1-TRAIN TRACKS. The concept of π_1-train tracks for a surface M is introduced in [1] where they are used, in conjunction with cutting sequences, to study simple loops on M. We briefly recall the definitions involved where the surface M in question is Σ, represented as the region $R \subset D$ as in section 2.1.

Definitions:

- A **π_1-train track** τ on R is a collection of pairwise disjoint arcs,
  $\alpha_j \colon [0,1] \to \bar{R}$, such that
  $\alpha_j(0) \in s(X_k)$,
  $\alpha_j(1) \in s(X_{k'})$,
  $\alpha_j(t) \in \operatorname{Int} R$, $t \in (0,1)$,
  X_k, $X_{k'} \in G$, where $s(X_k)$ and $s(X_{k'})$ are distinct sides of R, and such that at most one arc joins each pair of sides.

- A **weighting** μ on a π_1-train track τ is an assignment of a non-negative number $\mu(\alpha_j)$ to each arc α_j of τ. The weighting is **integral** if $\mu(\alpha_j)$ is an integer for each j.

3.2.1 Collapsing multiple loops

Definition: A multiple loop γ is said to be **supported** on a π_1-train track τ in R if each G-segment of $\tilde{\gamma}$ joins sides X, X' of R for which there is a branch $\alpha(X, X')$ in τ.

Let γ be a reduced multiple simple loop on Σ. Since γ is simple, its G-segments are a family of pairwise disjoint arcs in R. Since it reduced, each segment has endpoints on two distinct sides of R. We may, therefore, associate to γ a π_1-train track $\tau(\gamma)$: $\tau(\gamma)$ has an arc joining sides s, s' of R if and only if there is a G-segment $\tilde{\gamma}_i$ joining the same pair. Clearly, γ is supported on $\tau(\gamma)$.

We define an **integral weighting** $\mu(\gamma)$ of $\tau(\gamma)$ by assigning to the arc $\alpha(s, s')$ the number of segments of $\tilde{\gamma}_i$ joining s to s'.

3.2.2 Simple words

A word $W = E_1 \cdots E_r \in G$, $E_i \in G_0$, is **simple** if the corresponding homotopy class on Σ has a simple representative γ. Given a simple cyclically reduced word W, we can construct the weighted π_1-train track $(\tau(\gamma), \mu(\gamma))$ as described in section 3.2.1. A major advantage of π_1-train tracks, that is of central importance to us here, is that $(\tau(\gamma), \mu(\gamma))$ may also be constructed purely combinatorially from the word W as follows:

For $X, Y \in G_0$, $X \neq Y$, let

$$n(X,Y) = \#\{i \in \{1, \dots, r\} | E_i = X, E_{i+1} = Y, \text{ or } E_i = \bar{Y}, E_{i+1} = \bar{X}\}$$

where we define $E_{r+1} = E_1$. In other words, $n(X, Y)$ is the number of occurrences of either of the two consecutive pairs XY or $\bar{Y}\bar{X}$ in the cyclic word W. Using cutting sequences, it is not hard to see that $\tau(\gamma)$ will have a strand α in R joining $s(X)$ to $s(\bar{Y})$ whenever $n(X,Y) > 0$, and that $\mu(\gamma)(\alpha) = n(X,Y)$. For example, if $\gamma = \gamma^*$, the loop of the example in Figure 2 where $W = T\bar{S}_2\bar{S}_1T\bar{S}_2S_1\bar{T}S_1$, then $n(T,\bar{S}_1) = n(S_1,T) = 1$. The π_1-train track $(\tau(\gamma^*),\mu(\gamma^*))$ is shown in Figure 3.

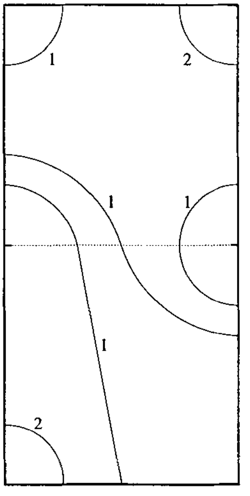


Figure 3. The π_1-train track $(\tau(\gamma^*),\mu(\gamma^*))$.

3.3 $\pi_{1,2}$-TRAIN TRACKS. We now adapt the notion of π_1-train tracks defined relative to the fixed fundamental region R to the context of the fundamental groupoid $\Gamma = \pi_{1,2}(\Sigma)$. To do this, we consider train tracks relative to each of the two halves R_1, R_2 of R defined in section 2.4.

Definitions:

- A $\pi_{1,2}$-**train track** τ is a collection of pairwise disjoint arcs $\alpha_{i,j}$, $i = 1,2$, $j = 1,\dots,n$ on R_1 and R_2, $\alpha_{i,j}$: $[0,1] \to \bar{R}_i$, such that $\alpha_{i,j}(0) \in s(y_k)$, $\alpha_{i,j}(1) \in s(y_{k'})$, $\alpha_{i,j}(t) \in \operatorname{Int} R_i$, $t \in (0,1)$, where $s(y_k)$ and $s(y_{k'})$ are distinct sides of R_i, and such that at most one arc joins each pair of sides.

- A **weighting** μ on a $\pi_{1,2}$-train track τ is an assignment of a non-negative number $\mu(\alpha_{i,j})$ to each arc $\alpha_{i,j}$ of τ. The weighting is **integral** if $\mu(\alpha_{i,j}) \in \mathbf{Z}^+$ for each j.

- An arc α of τ is called a **corner branch** if it joins two adjacent sides of R_1 or of R_2.

  Clearly, a train track on R_i can have at most four corner branches and one additional arc joining one of the two pairs of opposite sides.

Remark: This simple maximal configuration is the same for a once punctured torus and it is precisely this observation that makes the decomposition of R into R_1 and R_2 so useful.

- A multiple simple loop γ on Σ is said to be **supported** on a train track $\tau(\gamma)$ in R_i if each Γ-segment of γ joins sides s, s' of R_i for which there is a branch $\alpha(s, s')$ in τ.

The Γ-segments of a multiple simple reduced loop collapse to form a weighted $\pi_{1,2}$-train track $(\tau(\gamma), \mu(\gamma))$ which supports γ in exactly the same way as the G-segments collapse to form a π_1-train track. Similarly, we can collapse simple reduced Γ-words just as we collapsed G-words in section 3.2.2.

The $\pi_{1,2}$-train track associated to the loop γ^* of the example in section 2.2 is shown in Figure 4.

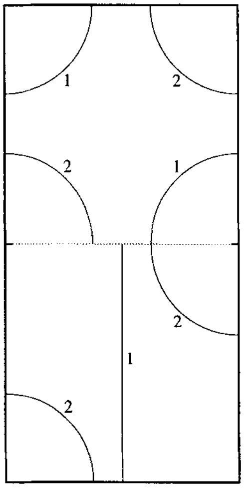


Figure 4. The $\pi_{1,2}$-train track for γ^* with Γ-word $w = t_1\bar{s}_2t_2\bar{s}_1t_1\bar{s}_2t_2s_1\bar{t}_2\bar{t}_1s_1$.

3.4 Proper Weightings. In [1], necessary and sufficient conditions are given on a π_1-train track to ensure that it comes from a multiple reduced simple loop γ. Here we carry out an analogous analysis of relations among weights for $\pi_{1,2}$-train tracks.

Observe that when R_1 and R_2 are projected to Σ their sides are identified in pairs; for example, $s(s_1)$ is paired to $s(\bar{s}_1)$ and $s(t_2)$ is paired to $s(\bar{t}_2)$. For each puncture v on Σ, there are exactly four corner branches cutting off those sides of R_1 and R_2 that end at v. These corner branches can be drawn so that the projections of their endpoints link to form a loop around v.

It is thus natural to impose the following conditions on weighted $\pi_{1,2}$- train tracks:

1. *Side pairing relations:* the sum of the weights of the arcs landing on a given side is the same as the sum of the weights of the arcs landing on the paired side.

2. *Boundary relations:* if a set of corner branches link to form a loop around a puncture v, then at least one of them has weight zero.

We call a weighting μ satisfying these conditions **proper**. We remark that the side pairing relations correspond to the switch conditions in Thurston's theory of train tracks, while the boundary conditions imply no components of the complement of the train track are punctured nullgons; see [9].

With only very minor modifications, we can apply Theorem 1.3 of [1] to get

THEOREM 3.1: *Let τ be a $\pi_{1,2}$-train track on R_1, R_2 with proper integral weighting μ. Then there is a unique (up to homotopy) multiple simple loop γ on Σ supported on τ with $\mu(\gamma) = \mu$.*

If the weighting μ comes from a simple cyclically reduced word W as in section 3.2.2, then γ is connected and represents the element W in $\pi_1(\Sigma)$.

The idea of the proof is as follows:

For each $\alpha(s, s')$ in τ with $n(s, s') \neq 0$, draw $n(s, s')$ strands joining s and s' in such a way that when all the strands for all the α's are drawn, they are disjoint. The crucial point is that this condition completely determines the **order** in which these strands meet each of the sides of ∂R_i. Hence there is a unique way to link up the strands to form a multiple simple loop γ on Σ. Since the boundary relations are satisfied, γ contains no loops around the punctures. In particular, if μ comes from a simple cyclically reduced word W, the arcs associated to the corresponding simple loop γ are laid down in the cyclic order determined by the word W as in 3.2.2; this is the same as the order determined by the line up of the strands so we recover the loop γ. ∎

Remark: The main significance of this theorem can be summarized as

- To a cyclically reduced simple word W we can combinatorially associate a weighted $\pi_{1,2}$-train track

  and

- The information contained in a weighted $\pi_{1,2}$-train track is sufficient to completely reconstruct W up to inversion and cyclic permutation.

This correspondence between weights, simple words and sequences encodes highly non-trivial information. It is this interplay that is the key to Theorem 5.3 below.

It is possible that, even if a loop γ is not simple, it may, by the collapsing process of 3.2.1, be supported on a π_1-train track τ. Counting the number of G-segments joining the different pairs of sides will in this case again give a weighting τ. (The side pairing conditions will automatically hold but the boundary conditions may be violated even if γ contains no boundary parallel loops.) The gluing-up process determined by the order of the strands on the sides of the R_i will rejoin the G-segments in a cyclic order and the resulting simple (possibly multiple) loop will be different from γ. This simple but fundamental and subtle observation is a key tool which provides an easy method to test whether or not a given word represents a simple loop.

In Figure 4, we have drawn the $\pi_{1,2}$-train track for the loop γ^*. The Γ-cutting sequence for γ^* gives us the word $w = t_1\bar{s}_2t_2\bar{s}_1t_1\bar{s}_2t_2s_1\bar{t}_2\bar{t}_1s_1$ for this train track. Contrast this with the G word $W = T\bar{S}_2\bar{S}_1T\bar{S}_2S_1\bar{T}S_1$ and the π_1-train track of Figure 3 that we found for the same loop in section 2.2.

The reader is encouraged to work out this example carefully in order to fully understand how the words and train tracks correspond.

4. $\pi_{1,2}$-Coordinates

We begin this section with a more detailed study of proper weightings. We show that the number of independent parameters in a proper weighting is four. We then associate to any proper weighting (equivalently, to any multiple simple loop γ or cyclically reduced simple word W) a set of four integers, called its $\pi_{1,2}$-coordinates, that completely determine both the underlying $\pi_{1,2}$-train track and the weightings on each branch.

4.1 Relations among weights. For convenience we draw the R_i's as squares and label the strands in them as in Figure 5 where $x_i, y_i, w_i, z_i, u_i \in \mathbf{Z}^+$, $i = 1, 2$. We refer to sides as horizontal, vertical, top, bottom, left, right relative to the configuration shown in Figure 5. Where convenient, we refer to R_i as a "box".

Definition: If the weights on all four corner strands in R_i are non-zero, we say that R_i **contains a cross.**

LEMMA 4.1: *At most one of the two boxes R_i contains a cross. In the other box, either $x_j = z_j = 0$ or $y_j = w_j = 0$.*

Proof: Any horizontal strand corresponds to a side pairing given by an S_i and thus contributes only to the side pairing relation associated to that S_i. Its weight appears on both sides of the equation for this relation and cancels out. Thus, we may suppose without loss of generality that one strand in each box is vertical; see Figure 5.

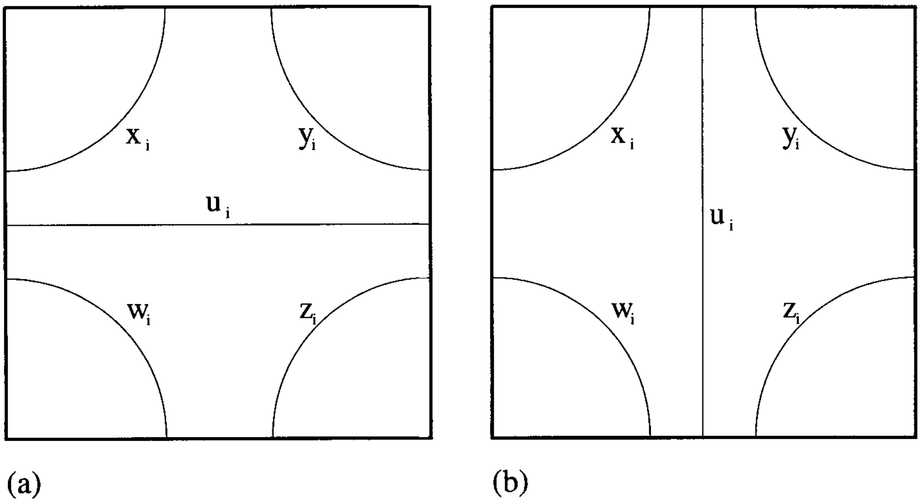


Figure 5. Configurations of a box with a cross.

The boundary relations imply that at least one of weights x_1, y_1, w_2, z_2 vanishes (see section 3.4). Similarly, one of w_1, z_1, y_2, x_2 vanishes. Assume that $z_2 = 0$. If $y_2 = 0$ also, then because of the S_2 side pairing in the bottom box, the weights on all the corner strands vanish and we are done. If $x_2 = 0$, opposite corner strands vanish in the bottom box and we are done. Assume, therefore, that $x_2, y_2 \neq 0$ and hence that either $w_1 = 0$ or $z_1 = 0$. The side pairings by the S_i give

$$x_1 + w_1 = y_1 + z_1, \tag{1}$$
$$y_2 = x_2 + w_2. \tag{2}$$

Using the T pairing and matching the two boxes across the middle give

$$x_1 + u_1 + y_1 = w_2 + u_2, \tag{3}$$
$$w_1 + z_1 + u_1 = x_2 + y_2 + u_2. \tag{4}$$

Substituting equations (2) and (3) in (4) we obtain

$$w_1 + z_1 = 2x_2 + x_1 + y_1. \tag{5}$$

Assume $w_1 = 0$. Then from equations (1) and (5) we get

$$x_2 = y_1 = 0.$$

On the other hand, if $z_1 = 0$, then from equations (1) and (5) we get

$$x_1 = x_2 = 0. \quad \blacksquare$$

COROLLARY 4.2: *The weights on the diagonally opposite corner strands in each R_i are equal:*

$$w_1 = y_1, \quad x_1 = z_1, \quad y_2 = w_2, \quad x_2 = z_2.$$

Proof: From the lemma above we may assume without loss of generality that $x_2 = z_2 = 0$. By the S_2 side pairing equation (2), $y_2 = w_2$. Now applying equations (1)–(4) we conclude $x_1 = z_1$ and then $y_1 = w_1$ as claimed. ∎

Remark 4.3: It follows immediately from this corollary that the total number of strands crossing the top and bottom sides of each box are equal. This is important below.

4.2 DEFINITION OF $\pi_{1,2}$-COORDINATES. From Corollary 4.2, we see easily that the number of independent parameters associated to a proper weighting is four. The $\pi_{1,2}$-coordinates

$$\mathbf{i}(\gamma) = (q_1(\gamma), p_1(\gamma), q_2(\gamma), p_2(\gamma)) \in (\mathbf{Z}^+ \times \mathbf{Z})^2$$

of a multiple simple loop γ are a convenient way of encoding this information in a form that is global in the sense that it is independent of the underlying configuration of strands; indeed (cf. Theorem 4.5 below), the strand configuration may be recovered directly from $\mathbf{i}(\gamma)$. The coordinates are also a convenient way of decoupling the information contained in the two boxes.

Definition: Let μ be a proper weighting on a $\pi_{1,2}$-train track τ. With the notation of Figure 5 we define

$$q_i(\gamma) = |x_i - y_i|, \tag{6}$$
$$p_i(\gamma) = \epsilon_i(u_i + |x_i - y_i|). \tag{7}$$

if the configuration is as in figure 5(a) and we define

$$q_i(\gamma) = u_i + |x_i - y_i|, \tag{8}$$
$$p_i(\gamma) = \epsilon_i|x_i - y_i|, \tag{9}$$

if it is as in Figure 5(b), where $\epsilon_i = 1$ if $x_i \geq y_i$ and $\epsilon_i = -1$ otherwise.

(Recall from Corollary 4.2 that $w_1 = y_1, x_1 = z_1, x_2 = z_2$ and $y_2 = w_2$.)

The motivation for these definitions is the following. Each R_i projects to a cylinder C_i contained in Σ, such that each boundary curve contains one of the punctures. For the moment, think of identifying the boundary curves on each of the cylinders C_i to obtain a torus $\mathbf{T}_i$. The torus $\mathbf{T}_i$ may equally well be thought of as being obtained by identifying the horizontal and vertical sides of R_i. By Remark 4.3, the numbers of Γ-segments of any multiple simple loop γ meeting the top and bottom side of each R_i are equal. Thus we may, in a natural way, identify the strands of γ lying in R_i to form a multiple loop γ_i on $\mathbf{T}_i$. We see that $\chi_i = \min(x_i, y_i)$ copies of each of the corner strands in R_i glue up to form χ_i loops around the vertex. The remaining strands glue to form a multiple of a curve homotopic to a straight line L in the plane $\mathbf{R}^2$ covering $\mathbf{T}_i$. It is easy to see that this line has slope q_i/p_i. Thus the sign of p_i is interpreted as the slope of the projection of γ_i on $\mathbf{T}_i$: the slope is positive if $x_i > y_i$ and negative if $x_i < y_i$. If $x_i = y_i$, L is either horizontal, $q_i = 0, p_i \neq 0$ and ϵ_i is chosen so that $p_i > 0$, or L is vertical and $q_i > 0, p_i = 0$.

We read off the $\pi_{1,2}$-coordinates for the loop γ^* with Γ word

$$w = t_1\bar{s}_2t_2\bar{s}_1t_1\bar{s}_2t_2s_1\bar{t}_2\bar{t}_1s_2$$

of Figure 4 as $\mathbf{i}(\gamma^*) = (1, 3, -1, -2)$.

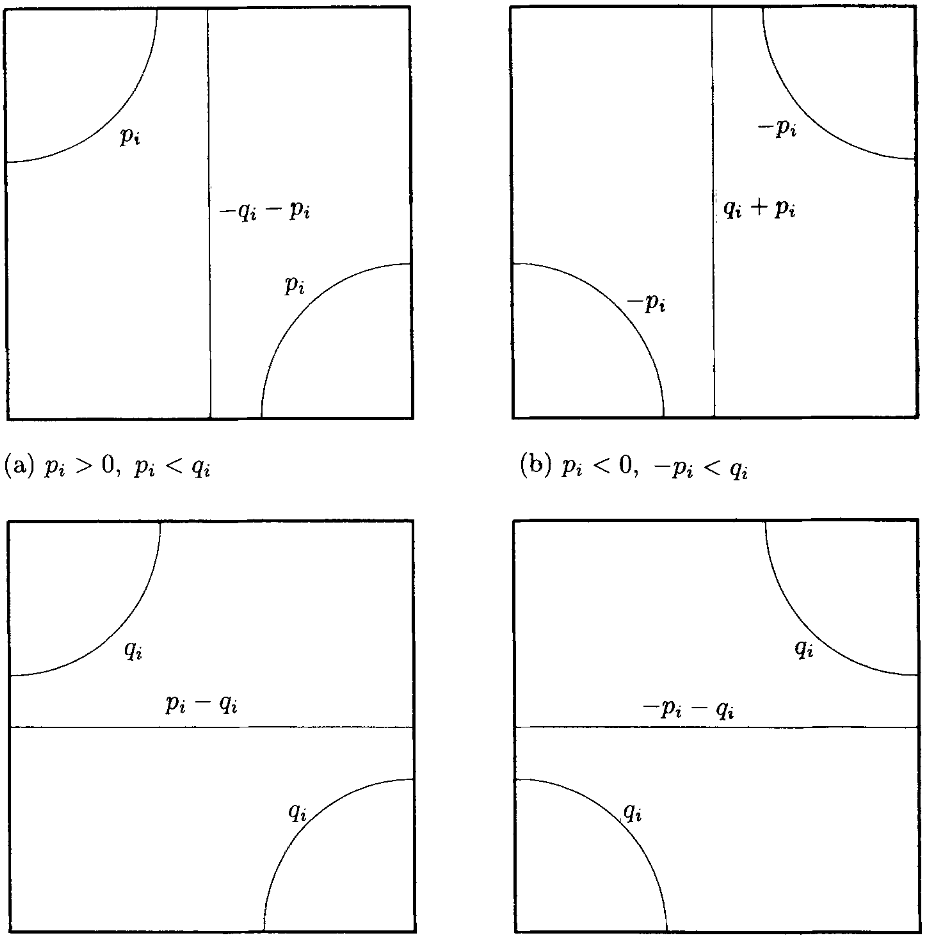
LEMMA 4.4: *For any multiple loop γ on Σ, $q_1(\gamma) - q_2(\gamma) \equiv 0 \bmod 2$.*

Proof: The union of the two curves s_1 and s_2 divides Σ into two components. This means that any other curve has algebraic intersection number zero with $s_1 \cup s_2$ and hence the geometric intersection number is zero (mod 2). ∎

THEOREM 4.5: *Any quadruple (q_1, p_1, q_2, p_2) with $q_i \in \mathbf{Z}^+, p_i \in \mathbf{Z}, i = 1, 2$, such that $q_1 - q_2 \equiv 0 \bmod 2$, $p_i \geq 0$ if $q_i = 0$ and ${q_1}^2 + {q_2}^2 + {p_1}^2 + {p_2}^2 \neq 0$, determines a unique proper weighted $\pi_{1,2}$-train track on R_1, R_2. Moreover, this correspondence is bijective.*

Proof: To prove the theorem we need only show how to reconstruct the train track configuration and the weights from (q_1, p_1, q_2, p_2).

First note that there are sixteen possible configurations as follows. In each box R_j we have a choice of one horizontal or one vertical strand. There is a cross in one of the two boxes R_1 or R_2 and in the other box we have the choice of which pair of diagonally opposite corners has zero weight. The four possible configurations in the box without the cross are shown in Figure 6.



(a) $p_i > 0,\ p_i < q_i$ (b) $p_i < 0,\ -p_i < q_i$

(c) $p_i > 0,\ p_i > q_i$ (d) $p_i < 0,\ -p_i > q_i$

Figure 6. Configurations in a box without the cross.

From Figure 6 it is clear that if R_i does not contain a cross, then the pair (p_i, q_i) fully determines the strand configuration and weights in R_i. If, on the other hand, R_i contains a cross, then the configuration and weights in R_i differ from those shown in Figure 6 by the addition of the weight $\chi = \min(x_i, y_i)$ to each of the four corner strands.

To fully determine the configuration it remains therefore to identify which box contains the cross and to find χ in terms of the (q_1, p_1, q_2, p_2). Since the sum of the weights meeting the horizontal sides agree, we see that if R_i is the box containing the cross then $2\chi + q_i = q_j$, where $j \neq i$. Thus $q_j \geq q_i$ and $\chi = (q_j - q_i)/2$. Hence we find that R_i contains a cross if and only if $q_i < q_j$. If $q_i = q_j$ for $i \neq j$ then neither box contains a cross. If $q_i \leq q_j$, then setting $\chi = (q_j - q_i)/2$ we obtain the entire configuration and weights as claimed. ∎

5. Patterns in simple words

The paper [10] contains a complete characterization of those reduced cutting sequences on a once punctured torus that come from *simple* loops. Up to cyclic permutation, such a sequence is fully determined by two intersection numbers completely analogous to the pair $(q_i(\gamma), p_i(\gamma))$ defined in section 4.2.

While not attempting a full solution of the analogous problem here, we proceed far enough in the same direction to be able to prove our main application, Theorem 6.1.

In this section, we discuss in some detail the patterns that occur in Γ-cutting sequences of simple words. As we shall see in Theorem 5.3, the possibilities for sequences corresponding to a simple reduced loop γ are intimately related to its $\pi_{1,2}$-coordinates $\mathbf{i}(\gamma)$.

We begin with an easy observation about the Γ-cutting sequence of an *arbitrary* reduced loop γ.

If $B = e_1 \cdots e_r$ and $B' = e'_1 \cdots e'_s$ where $e_i, e'_j \in \Gamma_0$ for $i = 1, \ldots, r$ and $j = 1, \ldots, s$, and if $e_r = e'_1$, then we call the word $e_1 \cdots e_r e'_2 \cdots e'_j$ the **amalgamation of** B **and** B' and write it as $B * B'$. We emphasize that the amalgamation of B and B' only has one copy of the common last letter of B and first letter of B'. This idea is motivated by the definition of a free product of two groups with amalgamation along a common subgroup.

PROPOSITION 5.1: *Let w be the Γ-cutting sequence of a reduced loop γ and assume γ is not homotopic to S_1 or S_2. Then up to cyclic permutation w is an amalgamation of blocks*

$$B_1 * D_1 * B_2 * D_2 * \cdots * B_r * D_r$$

where $B_i = a_i s_1^{k_i} b_i, D_i = b_i s_2^{l_i} a_i$, $k_i, l_i \in \mathbf{Z}$, $b_i \in \{\bar{t}_2, t_1\}$, $a_i \in \{\bar{t}_1, t_2\}$, $i = 1, \ldots, r$.

Proof: Let C_i be the projection of R_i onto Σ; each C_i is a cylinder. The trajectory of γ is divided into segments γ_j that are its successive intersections with the cylinders C_i. (Note that these segments are *not* the same as the segments of the G- or Γ-cutting sequences defined in section 2.2 or 2.5.) These segments γ_i lie alternately in C_1 and C_2 and have endpoints on the boundaries $C_1 \cap C_2$; that is, on the projections of the sides labelled $\{t_i, \bar{t}_i\}$, $i = 1, 2$. For each segment γ_j, we read its cutting sequence, including the labels of the sides containing both the initial and final points of γ_j. Thus, if $\gamma_j \subset C_1$, we have some sequences of the form $t_2 s_1^k t_1$, $k \in \mathbf{Z}$ and their inverses; these segments go into C_1 across one side and leave across the other. There are other sequences of the forms $t_2 s_1^l \bar{t}_2$ and $\bar{t}_1 s_1^l t_1$, $l \in \mathbf{Z}$; these sequences go into C_1 and out again across the same side. There are similar blocks for the segments $\gamma_j \subset C_2$. Amalgamating these blocks in an obvious way gives the result. ∎

If we specialize Proposition 5.1 to the case where γ is simple, we get much more information. We first deal with a very special case.

LEMMA 5.2: *Let γ be a reduced simple loop with $q_1 = q_2 = 0$, that is γ has $\pi_{1,2}$-coordinates $\mathbf{i}(\gamma) = (0, p_1, 0, p_2)$. Then γ is a loop homotopic to S_1 or S_2 and, moreover, p_2 or p_1 respectively is zero.*

Proof: By Lemma 4.1, neither box R_i contains a cross; hence it is easy to see that γ consists only of horizontal strands in each box. Since γ is connected on Σ, and since each horizontal strand closes up to form a loop on Σ homotopic to S_1 or S_2, the result follows. ∎

We now turn to our main result. For simplicity we make the statement in the case $q_2 \geq q_1; p_1, p_2 \geq 0; q_2 \neq 0$. If $q_2 < q_1$, we interchange the subscripts 1 and 2; if $p_j < 0$, we replace all occurrences of s_j with s_j^{-1}. The case $q_2 = 0$ is covered by the lemma above.

For $k \in \mathbf{Z}$, set

$$I_k = t_1 s_2^k t_2, \quad J_k = t_2 s_1^k t_1, \quad K = t_2 s_1 \bar{t}_2, \quad L = \bar{t}_1 s_1 t_1.$$

If $A = e_1 \cdots e_r$ is a block, write $\bar{A}$ or A^{-1} for the block $\bar{e}_r \cdots \bar{e}_1$. Also, set $\chi = |q_2 - q_1|/2$.

THEOREM 5.3: *Let w be the Γ-cutting sequence of a reduced simple loop γ with $\mathbf{i}(\gamma) = (q_1, p_1, q_2, p_2)$. Assume $q_2 \geq q_1; p_1, p_2 \geq 0$; $q_2 \neq 0$. Then (after possible cyclic permutation and possibly replacing w by w^{-1}) w is made up by amalgamating*

- *q_2 blocks $I_{m_1}, \ldots, I_{m_{q_2}}, q_2 \geq 0$;*
- *q_1 blocks $J_{n_1}, \ldots, J_{n_{q_1}}, q_1 \geq 0$;*
- *χ blocks K or $\bar{K}$ and χ blocks L or $\bar{L}$*

in sections $\mathcal{S}$, each of the form

$$\mathcal{S} = L^{\pm 1} * Q * K^{\pm 1} * \bar{Q}'$$

where Q and Q' are blocks of the form

$$I_{k_1} * J_{l_1} * I_{k_2} * \cdots * I_{k_r}, \quad r \geq 1, \; k_i, l_i \in \mathbf{Z}^+,$$
$$\{k_1, \ldots, k_r\} \subset \{m_1, \ldots, m_{q_2}\}, \quad \{l_1, \ldots, l_{r-1}\} \subset \{n_1, \ldots, n_{q_1}\}.$$

(If $r = 1$ there are no blocks J_{l_i}.) In w, the q_2 blocks I_{m_i} and q_1 blocks J_{n_j} are partitioned exactly among the terms Q and $\bar{Q}'$ in the sections $\mathcal{S}$. If $q_2 = q_1 \neq 0$ ($\chi = 0$), then there is only one section of the form Q. In all cases, $\sum_{j=1}^{q_2} m_j = p_2$ and $\sum_{j=1}^{q_1} n_j = p_1$.

Proof: First, we establish which types of blocks occur. The idea is to follow the method outlined in Proposition 5.1 watching the directions of the strands and how they are identified.

In the box R_2, since $p_2 \geq 0$, we are in the configuration given in Figure 6(a) or (c). Hence a segment of γ entering the cylinder C_2 across the bottom edge of R_2 must always contribute $t_1 s_2^m t_2$, $m \geq 0$ to the cutting sequence; in other words, a block I_m, $m \geq 0$. Blocks from strands entering R_2 across the top of R_2 are of the form $\bar{I}_m$. In the top box, R_1, we have the same configuration as in R_2, with an additional χ strands at each corner. Thus a segment of γ entering across the bottom of R_1 must either contribute $t_2 s_1^n t_1$, $n \geq 0$, if it leaves across the top of R_2, or $t_2 s_1 \bar{t}_2$ if it leaves across the bottom. This gives blocks J_n or $K, \bar{K}$. Similar reasoning for segments that enter across the top of R_1 gives blocks $\bar{J}_n$ (for segments leaving across the bottom) and $L, \bar{L}$ (for segments leaving across the top).

It remains to count the number of blocks of different types. The total number of strands in C_2 crossing the bottom of R_2 is q_2, and this is clearly the total number of blocks I_{m_j}. Since the occurrences of $s_2^{m_j}$ in blocks I_{m_j} all have positive exponents, while those in blocks $\bar{I}_{m_j}$ have negative exponents, it follows that the total exponent sum $\sum_{j=1}^{q_2} m_j$ is exactly the number of strands crossing the right vertical side of box R_2; namely p_2. In the top box R_1 it is clear that $\chi = (q_2 - q_1)/2$ strands from each of the bottom two corners link up to form χ blocks of type K or $\bar{K}$. Similarly, χ strands in each of the top two corners link into χ blocks of type L or $\bar{L}$. This deals with $2\chi = q_2 - q_1$ of the strands crossing each vertical side of R_1. All other strands crossing the vertical sides of R_1 must link in such a way that they enter across the bottom and leave across the top, or vice versa, forming blocks of type J_{n_j} or $\bar{J}_{n_j}$. The total number of strands meeting a vertical side of R_1 is $2\chi + p_1$. Hence we find $\sum_{j=1}^{q_1} n_j = p_1$.

Now we establish the order in which these blocks are amalgamated to form w.

All of the blocks that occur are of the form $A = xs^m y$, $x, y \in \{t_i, \bar{t}_i\}$, $s \in \{s_i, \bar{s}_i\}$, $i = 1, 2$, $m \in \mathbf{Z}$. Let $A = xs^m y$, $A' = x'(s')^n y'$ be two such blocks. Since we have included labels for both the initial and final points, it is clear that in the Γ-cutting sequence for w, **A** can be followed by **A**$'$ if and only if $y = x'$. In w, these two blocks contribute a term $A * A'$. We call the patterns forced by these conditions **adjacency rules**.

Applying the adjacency rules applied to the blocks above we see that I can be followed by J or $K^{\pm 1}$ and J can be followed only by I. An $L^{\pm 1}$ can be followed only by I and $K^{\pm 1}$ can be followed only by $\bar{I}$. Likewise $\bar{I}$ can be followed by $\bar{J}$ or $L^{\pm 1}$ and $\bar{J}$ can only be followed by $\bar{I}$.

Now suppose $\chi > 0$. We have shown above that w contains the block L or $\bar{L}$. The order of the blocks is completely determined by the rules above and we deduce the complete pattern of the section $\mathcal{S}$. Notice that once an $L^{\pm 1}$ has occurred, $K^{\pm 1}$ is forced later in the section in order for the curve to close up. If $\chi = 0$, so that there is no $L^{\pm 1}$, then there is also no $K^{\pm 1}$ and w is of the form Q or $\bar{Q}$. ∎

Example: The block decomposition for the loop γ^* from Figure 4 with word $w = t_1\bar{s}_2t_2\bar{s}_1t_1\bar{s}_2t_2s_1\bar{t}_2\bar{t}_1s_1$ (where $p_1 < 0, p_2 < 0$) is

$$I_{-1} * J_{-1} * I_{-1} * K * I_0 * L.$$

6. Application to trace polynomials

6.1 STATEMENT OF THE THEOREM. In this section, we apply Theorem 5.3 to study the traces of the matrices in a specific family of representations $\rho\colon \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{C})$ that correspond to simple closed curves on Σ.

For $\sigma, \tau \in \mathbf{C}^2$ we define

$$\rho_{\sigma,\tau}(S_1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \rho_{\sigma,\tau}(S_2) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \rho_{\sigma,\tau}(T) = \begin{pmatrix} 1+\sigma\tau & \sigma \\ \tau & 1 \end{pmatrix}.$$

Since $\pi_1(\Sigma)$ is a free group on the generators S_1, S_2, T this defines a representation

$$\rho = \rho_{\sigma,\tau}\colon \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbf{C}).$$

If $g \in \pi_1(\Sigma)$, then the matrix coefficients and hence the trace of $\rho(g)$ are clearly polynomials in σ and τ. The main result of this section is

THEOREM 6.1: *Let γ be a simple closed curve on Σ with $\mathbf{i}(\gamma) = (q_1, p_1, q_2, p_2)$ where $q_1 + q_2 \neq 0$. Let γ be represented by $g \in \pi_1(\Sigma)$. Then $\operatorname{Tr} \rho_{\sigma,\tau}(g)$ is a polynomial in σ and τ of degree q_1 in σ and q_2 in τ whose top terms have the following form:*

$$\begin{aligned}\operatorname{Tr} \rho(g) &= \pm 2^{|q_2 - q_1|}\big(\sigma^{q_1}\tau^{q_2} + 2p_1\sigma^{q_1-1}\tau^{q_2} + 2p_2\sigma^{q_1}\tau^{q_2-1}\big) + O\big(q_1 + q_2 - 2\big) \\ &= \pm 2^{|q_2 - q_1|}\big(\sigma + 2p_1/q_1\big)^{q_1}\big(\tau + 2p_2/q_2\big)^{q_2} + O\big(q_1 + q_2 - 2\big),\end{aligned}$$

where the notation $O(q_1 + q_2 - 2)$ means terms of degree at most q_1 in σ and q_2 in τ but with total degree in σ and τ at most $q_1 + q_2 - 2$.

Note that if $q_1 + q_2 = 0$ then $\rho(g)$ is parabolic; compare Lemma 5.2. If $q_1 = 0$ then $\operatorname{Tr} \rho(g)$ is a function of τ alone, and likewise if $q_2 = 0$ then $\operatorname{Tr} \rho(g)$ is a function of σ alone.

6.2 MOTIVATION. Although not necessary for what follows, we would like to present some motivation for the above result. We discovered Theorem 6.1 in the course of our investigations of the Maskit embedding for the Teichmüller space of Σ.

For suitable values of $\sigma, \tau \in \mathbf{C}^2$, one can show, using Maskit's second combination theorem, that $G = G(\sigma, \tau) = \rho_{\sigma,\tau}(\pi_1(\Sigma))$ is a Kleinian group. Its regular set $\Omega(G)$ contains exactly one simply connected invariant component Ω^* with quotient $\Omega^*/G = \Sigma$ and infinitely many non-invariant components that are round disks with quotient two triply punctured spheres (see [7]). By techniques analogous to those in [6], one can show that the set

$$\mathcal{M} = \{(\sigma, \tau) \in \mathbf{C}^2 | \Im\sigma, \Im\tau > 0, \Omega(G(\sigma, \tau)) \text{ has the above properties }\}$$

is an embedding, called the **Maskit embedding**, of the Teichmüller space of Σ.

In [6] we gave a detailed analysis of $\mathcal{M}_1$, the corresponding Maskit embedding of a once punctured torus Σ_1, using the method of pleating coordinates. Key to that analysis was a result ([6], proposition 3.1) analogous to Theorem 6.1, about the trace polynomials of elements representing simple closed curves on Σ_1. In [7] we use ideas similar to those in [6] to give a detailed analysis of the structure and boundary of $\mathcal{M}$ in which Theorem 6.1 plays a similarly central role.

An analogous discussion and theorem for trace polynomials in the Maskit embedding of the Teichmüller space of four and five times punctured spheres is presented in [4].

6.3 PROOF OF THEOREM 6.1: OUTLINE. Before giving details of the proof of Theorem 6.1 we sketch an outline. First we factor $\rho_{\sigma,\tau}(T)$ as $M_1(\sigma)M_2(\tau)$ in such a way that

$$\psi_{\sigma,\tau}(s_i) = \rho_{\sigma,\tau}(S_i), \quad \psi_{\sigma,\tau}(t_1) = M_1(\sigma), \quad \psi_{\sigma,\tau}(t_2) = M_2(\tau)$$

defines a representation $\psi_{\sigma,\tau}$ of the fundamental groupoid $\Gamma = \pi_{1,2}(\Sigma)$ into $\mathrm{SL}(2,\mathbf{C})$. Next, we compare G- and Γ-cutting sequences enabling us to express $\rho_{\sigma,\tau}(\gamma)$ as a product

$$\prod_{i=1}^{N} \begin{pmatrix} 1 & \alpha_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta_i & 1 \end{pmatrix}$$

where the α_i (resp. β_i) are simple linear functions of σ (resp. τ). Finally, we use a general result about matrix products of this form to obtain the result.

We remark that a much simpler version of this idea, where the factorization of $\rho_{\sigma,\tau}(T)$ is not necessary, may be used to give an alternative proof of proposition 3.1 of [6].

6.4 MATRIX PRODUCTS. For notational purposes we define a family of polynomials ϕ_n in n variables inductively as follows:

$$\phi_0 = 1, \quad \phi_1(x_1) = x_1,$$
$$\phi_{n+1}(x_1, \ldots, x_{n+1}) = \phi_n(x_1, \ldots, x_n)x_{n+1} + \phi_{n-1}(x_1, \ldots, x_{n-1}).$$

These polynomials are the continuant polynomials known to Euler (see, for example, p. 288 of [5]). Applying the definition twice we see that

$$\begin{aligned}\phi_{n+2}(x_1, \ldots, x_{n+2}) &= \phi_{n+1}(x_1, \ldots, x_{n+1})x_{n+2} + \phi_n(x_1, \ldots, x_n) \\ &= \phi_n(x_1, \ldots, x_n)(x_{n+1}x_{n+2} + 1) + \phi_{n-1}(x_1, \ldots, x_{n-1})x_{n+2}.\end{aligned}$$

It is easy to see that $\phi_n(x_1, \ldots, x_n)$ is a sum of the following form. The first term is the product $P = x_1 \cdots x_n$; the next $n-1$ terms are $P/(x_j x_{j+1})$ as j varies from 1 to $n-1$. The remaining terms are obtained by dividing P by more adjacent pairs; all we will need about them is that they are products of distinct x_k's, $k \in \{0, \ldots, n\}$, such that if n is even there are at most $n/2 - 2$ factors with k even and the same number with k odd while if n is odd there are $(n-1)/2$ factors with k even and $(n+1)/2$ factors with k odd. Thus, up to lower order terms $\phi_n(x_1, \ldots, x_n)$ is

$$x_1x_2 \cdots x_{n-1}x_n + (x_1x_2 \cdots x_{n-3}x_{n-2} + x_1x_2 \cdots x_{n-3}x_n + \cdots + x_3x_4 \cdots x_{n-1}x_n).$$

With this notation,

Proposition 6.2:

$$\prod_{j=1}^{k}\begin{pmatrix} 1 & \alpha_j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta_j & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \phi_{2k}(\alpha_1,\beta_1,\ldots,\alpha_k,\beta_k) & \phi_{2k-1}(\alpha_1,\beta_1,\ldots,\beta_{k-1},\alpha_k) \\ \phi_{2k-1}(\beta_1,\alpha_2,\ldots,\alpha_k,\beta_k) & \phi_{2k-2}(\beta_1,\alpha_2,\ldots,\beta_{k-1},\alpha_k) \end{pmatrix}.$$

Proof: The proof is by induction. It is easy to see that

$$\begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta_2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+\alpha_1\beta_1 & \alpha_1 \\ \beta_1 & 1 \end{pmatrix}\begin{pmatrix} 1+\alpha_2\beta_2 & \alpha_2 \\ \beta_2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_1\beta_1\alpha_2\beta_2+\alpha_2\beta_2+\alpha_1\beta_2+\alpha_1\beta_1+1 & \alpha_1\beta_1\alpha_2+\alpha_2+\alpha_1 \\ \beta_1\alpha_2\beta_2+\beta_2+\beta_1 & \beta_1\alpha_2+1 \end{pmatrix}$$
$$= \begin{pmatrix} \phi_4(\alpha_1,\beta_1,\alpha_2,\beta_2) & \phi_3(\alpha_1,\beta_1,\alpha_2) \\ \phi_3(\beta_1,\alpha_2,\beta_2) & \phi_2(\beta_1,\alpha_2) \end{pmatrix}.$$

Now assume that the kth product has the required form. The $(k+1)$th product is

$$\begin{pmatrix} \phi_{2k}(\alpha_1,\ldots,\beta_k) & \phi_{2k-1}(\alpha_1,\ldots,\alpha_k) \\ \phi_{2k-1}(\beta_1,\ldots,\beta_k) & \phi_{2k-2}(\beta_1,\ldots,\alpha_k) \end{pmatrix}\begin{pmatrix} 1+\alpha_{k+1}\beta_{k+1} & \alpha_{k+1} \\ \beta_{k+1} & 1 \end{pmatrix}.$$

The top left-hand entry of this product is

$$\phi_{2k}(\alpha_1,\beta_1,\ldots,\alpha_k,\beta_k)(\alpha_{k+1}\beta_{k+1}+1)+\phi_{2k-1}(\alpha_1,\beta_1,\ldots,\beta_{k-1},\alpha_k)\beta_{k+1}$$
$$=\phi_{2k+2}(\alpha_1,\beta_1,\ldots,\alpha_{k+1},\beta_{k+1}).$$

The top right-hand entry is

$$\phi_{2k}(\alpha_1,\beta_1,\ldots,\alpha_k,\beta_k)\alpha_{k+1}+\phi_{2k-1}(\alpha_1,\beta_1,\ldots,\beta_{k-1},\alpha_k)$$
$$=\phi_{2k+1}(\alpha_1,\beta_1,\ldots,\beta_k,\alpha_{k+1}).$$

The bottom left-hand entry is

$$\phi_{2k-1}(\beta_1,\alpha_2,\ldots,\alpha_k,\beta_k)(\alpha_{k+1}\beta_{k+1}+1)+\phi_{2k-2}(\beta_1,\alpha_2,\ldots,\beta_{k-1},\alpha_k)\beta_{k+1}$$
$$=\phi_{2k+1}(\beta_1,\alpha_2,\ldots,\alpha_{k+1},\beta_{k+1}).$$

Finally, the bottom right-hand entry is

$$\phi_{2k-1}(\beta_1,\alpha_2,\ldots,\alpha_k,\beta_k)\alpha_{k+1}+\phi_{2k-2}(\beta_1,\alpha_2,\ldots,\beta_{k-1},\alpha_k)$$
$$=\phi_{2k}(\beta_1,\alpha_2,\ldots,\beta_k,\alpha_{k+1}).$$

This means that the $(k+1)$th product has the form

$$\begin{pmatrix} \phi_{2k+2}(\alpha_1,\beta_1,\ldots,\alpha_{k+1},\beta_{k+1}) & \phi_{2k+1}(\alpha_1,\beta_1,\ldots,\beta_k,\alpha_{k+1}) \\ \phi_{2k+1}(\beta_1,\alpha_2,\ldots,\alpha_{k+1},\beta_{k+1}) & \phi_{2k}(\beta_1,\alpha_2,\ldots,\beta_k,\alpha_{k+1}) \end{pmatrix}$$

as required. ∎

The following immediate corollary will be useful to us later.

COROLLARY 6.3: *The trace of*

$$\prod_{j=1}^{k}\begin{pmatrix} 1 & \alpha_j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta_j & 1 \end{pmatrix}$$

equals

$$\begin{aligned}&\phi_{2k}(\alpha_1,\beta_1,\ldots,\alpha_k,\beta_k)+\phi_{2k-2}(\beta_1,\alpha_2,\ldots,\beta_{k-1},\alpha_k)\\ &=\alpha_1\beta_1\cdots\alpha_k\beta_k+\alpha_1\beta_1\cdots\alpha_{k-1}\beta_{k-1}+\alpha_1\beta_1\cdots\alpha_{k-1}\beta_k\\ &\quad+\cdots+\alpha_2\beta_2\cdots\alpha_k\beta_k+\beta_1\alpha_2\cdots\beta_{k-1}\alpha_k+O(k-2,k-2),\end{aligned}$$

where $O(k-2,k-2)$ means products of at most $k-2$ of the α_j's and $k-2$ of the β_j's.

Notation: We remark that we are giving the notation $O(j,k)$ above and $O(r)$ in Theorem 6.1 two different meanings. Their use will depend on whether we are talking about expressions in α,β or σ,τ respectively. It will always be clear from the context what we mean.

6.5 THE FACTORIZATION. We now turn to the factorization of $\rho_{\sigma,\tau}(T)$. The idea is to extend the representation $\rho_{\sigma,\tau}$ to a representation $\psi_{\sigma,\tau}$ of $\pi_{1,2}(\Sigma)$ in such a way that we can exploit the patterns we found in Theorem 5.3 for Γ-sequences of simple closed curves.

Definitions:

- A **representation** ψ of $\Gamma=\pi_{1,2}(\Sigma)$ to $\mathrm{SL}(2,\mathbf{C})$ is a map $\psi\colon\Gamma\to\mathrm{SL}(2,\mathbf{C})$ such that $\psi(\gamma_1\gamma_2)=\psi(\gamma_1)\psi(\gamma_2)$ whenever $\gamma_1\gamma_2$ is defined in Γ.
- The representation ψ is **compatible** with ρ if $\psi(\gamma)=\rho(\gamma)$ whenever γ is a loop based at p_1. In particular,

$$\psi(s_1)=\rho(S_1),\quad \psi(t_1t_2)=\rho(T),\quad \psi(\bar t_2s_2t_2)=\rho(S_2)$$

  in accordance with the relations given by relations (**) in section 2.6.

PROPOSITION 6.4: *Let γ be a curve on Σ and let $w = e_1 \cdots e_r$, $W = E_1 \cdots E_s$ be its Γ- and G-cutting sequences respectively, $e_i \in \Gamma_0, E_j \in G_0$. Let ρ, ψ be compatible representations of Γ, G respectively in* $\mathrm{SL}(2, \mathbf{C})$. *Then*

$$\psi(w) = \psi(e_1) \cdots \psi(e_r), \quad \rho(W) = \rho(E_1) \cdots \rho(E_s) \quad \text{and} \quad \psi(w) = \rho(W).$$

Proof: This is clear from the definitions if we note that we may, after cyclic permutation or conjugation if necessary, make w a product of loops all starting at p_1. ∎

We also have

LEMMA 6.5: *Let ρ: $G \to \mathrm{SL}(2, \mathbf{C})$ be a representation of G and suppose $M_1, M_2 \in \mathrm{SL}(2, \mathbf{C})$ such that $M_1 M_2 = \rho(T)$, $M_2 \rho(S_2) = \rho(S_2) M_2$. Set $\psi(s_i) = \rho(S_i), \psi(t_i) = M_i$. For $\gamma \in \Gamma$ with Γ-cutting sequence $e_1 \cdots e_r$, $e_i \in \Gamma_0$, set $\psi(\gamma) = \prod_1^r \psi(e_i)$. Then ψ is a representation of Γ compatible with G.*

Proof: Since Γ is a free groupoid, the definition of ψ makes sense. The compatibility with ρ is a consequence of the relations $(**)$. ∎

COROLLARY 6.6: *Let $\rho = \rho_{\sigma,\tau}$ be the representation of $G = \pi_1(\Sigma)$ defined in section 6.1. Then*

$$\psi_{\sigma,\tau}(s_1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \psi_{\sigma,\tau}(s_2) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

and

$$\psi_{\sigma,\tau}(t_1) = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}, \quad \psi_{\sigma,\tau}(t_2) = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}$$

defines a representation of Γ into $\mathrm{SL}(2, \mathbf{C})$ *compatible with ρ.*

Proof: We have only to check that the conditions of Proposition 6.4 are satisfied. Note that, in addition, $\psi_{\sigma,\tau}(s_i)$ commutes with $\psi_{\sigma,\tau}(t_i)$. ∎

For $\xi \in \mathbf{C}$, set

$$M_1(\xi) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}, \quad M_2(\xi) = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix},$$

so that in the representation in Corollary 6.6

$$\psi_{\sigma,\tau}(s_1) = M_1(2), \quad \psi_{\sigma,\tau}(t_1) = M_1(\sigma), \quad \psi_{\sigma,\tau}(s_2) = M_2(2)\psi_{\sigma,\tau}(t_2) = M_2(\tau).$$

Note that $M_i(\xi)$ commutes with $M_i(\eta)$ for all $\xi, \eta \in \mathbf{C}$ and $M_i^{-1}(\xi) = M_i(-\xi)$, $i = 1, 2$. ∎

LEMMA 6.7: *Let γ be a simple closed curve with $\mathbf{i}(\gamma) = (q_1, p_1, q_2, p_2)$ and suppose that γ is represented by $g \in G$. Suppose $q_2 \geq q_1; p_1, p_2 \geq 0, q_2 \neq 0$. Then $\rho_{\sigma,\tau}(g)$ is a product of blocks, each of the form*

$$M_1(\pm 2)M_2(\tau + 2k_1)M_1(\sigma + 2l_1)\cdots$$
$$\cdots M_2(\tau + 2k_r)M_1(\pm 2)M_2(-\tau - 2k_{r+1})M_1(-\sigma - 2l_{r+1})\cdots M_2(-\tau - 2k_s).$$

The terms and order in this product correspond exactly to the blocks described in Theorem 5.3. (Note that the terms involving l_r and l_s are missing.)

Proof: The proof is based on Lemma 6.5. It is clearly enough to consider the image under ρ of one section $\mathcal{S} = L * Q * K * \bar{Q}'$ in the cutting sequence of γ. Amalgamating blocks and regrouping, using the definitions in section 5., we see that $\mathcal{S}$ can be rewritten as

$$(\bar{t_1} s_1^{\pm 1} t_1)(s_2^{k_1} t_2)(s_1^{l_1} t_1)(s_2^{k_2} t_2)\cdots(s_2^{k_r} t_2)(s_1)^{\pm 1}(\bar{t_2}\bar{s}_2^{k_{r+1}})(\bar{t_1}\bar{s}_1^{l_{r+1}})\cdots(\bar{t_2}\bar{s}_2^{k_s})$$

where $k_i, l_i \geq 0$ and terms with l_r and l_s are missing.

Now note that

$$\psi(\bar{t}_1 s_1 t_1) = \psi(s_1) = M_1(2),$$
$$\psi(s_2^k t_2) = M_2(2k)M_2(\tau) = M_2(\tau + 2k),$$
$$\psi(s_1^l t_1) = M_1(\sigma + 2l).$$

Using the count in Theorem 5.3 the result follows. ∎

We are finally ready to prove Theorem 6.1.

6.6 PROOF OF THEOREM 6.1. If $q_1 = q_2 = 0$ the result is trivial. Thus, without loss of generality, we may as usual assume $q_2 \geq q_1; p_1, p_2 \geq 0, q_2 \neq 0$. We leave the other cases to the reader.

We apply Proposition 6.2 to $\rho_{\sigma,\tau}(g) = \psi(g)$ using Lemma 6.7. Thus we set $k = q_2$ and $\beta_j = \pm(\tau + 2m_j)$, $j = 1, \ldots, q_2$; q_1 of the α_j are $\pm(\sigma + 2n_j)$ and the remaining 2χ are ± 2. The order of the terms is given by the block structure of Theorem 5.3.

By Corollary 6.3,

$$\begin{aligned}&\mathrm{Tr}\big(\rho_{\sigma,\tau}(g)\big)\\ &= \alpha_1\beta_1\cdots\alpha_k\beta_k + \alpha_1\beta_1\cdots\alpha_{k-1}\beta_{k-1} + \alpha_1\beta_1\cdots\alpha_{k-1}\beta_k\\ &\quad + \cdots + \alpha_2\beta_2\cdots\alpha_k\beta_k + \beta_1\alpha_2\cdots\beta_{k-1}\alpha_k + O(k-2, k-2).\end{aligned}$$

First consider the term $\alpha_1\beta_1\cdots\alpha_k\beta_k$. Clearly, this is of degree q_1 in σ and q_2 in τ. We have

$$\begin{aligned}
&\alpha_1\beta_1\cdots\alpha_k\beta_k\\
&= \pm 2^{2\chi}\prod_{j=1}^{q_1}(\sigma+2n_j)\prod_{j=1}^{q_2}(\tau+2m_j)\\
&\quad \pm 2^{2\chi}\left[\sigma^{q_1}\tau^{q_2}+2\sigma^{q_1}\tau^{q_2-1}\sum_1^{q_2} m_j+2\sigma^{q_1-1}\tau^{q_2}\sum_1^{q_1} n_j+O(q_1+q_2-2)\right].
\end{aligned}$$

From Theorem 5.3 we have $\sum m_j = p_2, \sum n_j = p_1$, hence this can be written

$$\pm 2^{2\chi}\left(\sigma+2\frac{p_1}{q_1}\right)^{q_1}\left(\tau+2\frac{p_2}{q_2}\right)^{q_2}+O(q_1+q_2-2).$$

Now consider the terms

$$\begin{aligned}
&\alpha_1\beta_1\cdots\alpha_{k-1}\beta_{k-1}+\alpha_1\beta_1\cdots\alpha_{k-1}\beta_k\\
&\quad+\cdots+\alpha_2\beta_2\cdots\alpha_k\beta_k+\beta_1\alpha_2\cdots\beta_{k-1}\alpha_k.
\end{aligned}$$

Each of these products involves dropping two consecutive terms, read cyclically, from a product

$$\pm 2(\tau+2m_j)(\sigma+2n_j)\cdots(\tau+2m_r)(\pm 2)(-\tau-2m_{r+1})(-\sigma-2n_{r+2})\cdots(-\tau-2m_s).$$

Dropping a pair involving σ and τ lowers the degree by 2. Thus we only need consider the effect of dropping consecutive terms when one of the pair is ± 2. Recall that these come from terms $L^{\pm 1}$ or $K^{\pm 1}$.

Writing ± 2 and the adjacent terms, we have

$$\cdots(\tau+2n_j)(\pm 2)(-\tau-2n_{j+1})\cdots.$$

Since we only have to consider terms of degree greater than q_1+q_2-2, we see that the terms of degree q_1+q_2-1 obtained by dropping each of the pairs

$$\pm(\tau+2n_j)(\pm 2) \quad\text{and}\quad (\pm 2)(-\tau-2n_{j+1})$$

cancel.

In the remaining terms, denoted by $O(k-2,k-2)$ in the expression for the trace in Corollary 6.3, the degree drops by at least 2 in τ. Combining all of these considerations gives the result. ∎

7. Measured lamination space

In this section we show how $\pi_{1,2}$-coordinates relate nicely to the standard Thurston theory of projective measured laminations. The work in this section is independent of sections 5 and 6.

7.1 BASIC THEORY. We begin by recalling some of the basic theory that we shall need.

Definitions: See e.g. [9].

- A **geodesic lamination** Λ on Σ is a closed subset $\Lambda \subset \Sigma$ which is a union of pairwise disjoint complete simple geodesics on Σ called its **leaves**.
- A **transverse measure** on Λ is an assignment of a measure to each arc transverse to the leaves of Λ that is invariant under the push forward maps along the leaves of Λ.
- A **measured geodesic lamination** on Σ is a geodesic lamination together with a transverse measure; the collection of all measured geodesic laminations on Σ is denoted by $\mathrm{ML}(\Sigma)$.

Let γ be a simple closed curve on Σ. The geodesic freely homotopic to γ is a special case of a geodesic lamination and we may define a transverse measure δ_γ as follows. To each transverse arc α and each measurable subset $E \subset \alpha$ assign to E the number of intersections of E with γ. This definition extends easily to multiple simple loops: if $m\gamma + m'\gamma'$ is such a loop, where $m, m' \in \mathbf{Z}^+$ are the respective multiplicities, the leaves of the corresponding geodesic lamination Λ are the disjoint geodesics freely homotopic to the loops γ, γ' and the transverse measure is $m\delta_\gamma + m'\delta_{\gamma'}$.

More generally, we may clearly form $\alpha\delta_\gamma + \alpha'\delta_{\gamma'}$ with $\alpha, \alpha' \in \mathbf{R}^+$. Such measured laminations are called **rational**: laminations of this kind are our main concern here.

Let

$$V = \left\{(q_1, p_1, q_2, p_2) \in \mathbf{Z}^4 - \{(0,0,0,0)\} \,\middle|\, q_1 \equiv q_2 \bmod 2\right\} / \approx$$

where $\approx$ is the equivalence relation

$$(q_1, p_1, q_2, p_2) \approx (-q_1, -p_1, q_2, p_2) \approx (q_1, p_1, -q_2, -p_2).$$

The choices we made in 4.2 of $q_i \geq 0$ and $p_i \geq 0$ if $q_i = 0$ clearly determine one element in each equivalence class of $\approx$. The result of Theorem 4.5 is equivalent to showing that there is a bijection between V and the space of homotopy classes of multiple simple loops on Σ. Let

$$\hat{V} = \{(q_1, p_1, q_2, p_2)\} \in \mathbf{R}^4 - \{(0,0,0,0)\}\}/\approx$$

where $\approx$ is the obvious extension of the relation defining V. Using Theorem 4.5 we see easily that the map $\mathbf{i}$, that assigns to a simple closed geodesic γ, the image of its $\pi_{1,2}$-coordinates $\mathbf{i}(\gamma)$ in V extends by linearity to a map from the space of rational measured laminations onto a dense subset of $\hat{V}$ containing all rational points.

THEOREM 7.1: *The map* $\mathbf{i}$ *extends to a homeomorphism* $\mathrm{ML}(\Sigma) \to \hat{V}$.

Proof: The essence of this theorem is well known so we give only a sketch here (see e.g. [3, 9, 12]). Suppose $\beta \in \mathrm{ML}(\Sigma)$ with underlying lamination Λ. Since all the leaves of Λ are disjoint and geodesic, their lifts intersect the regions R_i in segments forming one of the configurations (a) or (b) of Figure 5. This gives a $\pi_{1,2}$-train track. The weight assigned to each branch joining sides s, s' of R_i is the transverse measure of an arc transverse to all segments of Λ joining sides s to s' and meeting no other segments. From these weights, the numbers $q_i(\beta), p_i(\beta)$ can be computed as in Theorem 4.5.

Conversely, given $(q_1, q_2, p_1, p_2) \in \hat{V}$, we use the inequalities among the q_i and p_i to identify one of the sixteen strand configurations derived from Figures 5 and 6 just as in the proof of Theorem 4.5. This gives a proper weighting on the corresponding $\pi_{1,2}$-train track. Now we use the method described in [3, 9, 12] (the "highway picture") to construct a geodesic lamination with the given weights. By using π_1- and $\pi_{1,2}$-train tracks we eliminate all the considerable technicalities about ϵ-curvature of train tracks, bigons, transverse recurrence etc. in [9]. The essential point is that a G- or Γ-cutting sequence automatically defines a unique geodesic in $\mathbf{H}^2$.

Continuity is clear from the definition of the topology on PML (see, for example, [9]). ∎

7.2 THE PIECEWISE LINEAR CONE STRUCTURE OF ML. According to Thurston [12, 13, 9], the space ML of measured laminations on any hyperbolic surface has a piecewise linear positive cone structure. This means that ML consists of a finite number of cells, each of which is a positive cone in a vector space. This structure is clearly visible in $\hat{V}$: the cells are precisely the regions defined by the sixteen different configurations in Theorem 4.5. Two points in V lying in the same cell are supported on the same $\pi_{1,2}$-train track and thus we can sensibly add positive linear combinations of the weights.

The cells fit nicely along their boundaries; for example, the boundary between the partial configurations in Figure 5(a) and (b) is $p_1 = 0$.

7.3 PROJECTIVE LAMINATIONS AND S^3. The Thurston theory is mainly concerned with the space of **projective measured laminations** PML(Σ), which is the quotient of ML(Σ) by the relation $\beta \sim t\beta$, $t \in \mathbf{R}^+$. In our case this means that PML(Σ) $= \hat{V}/\mathbf{R}^+$, where the equivalence relation is induced by the diagonal multiplication of $\mathbf{R}^+$ on the coordinates.

According to Thurston, for a surface Σ of genus g and b punctures with $3g - 3 + b > 0$, PML(Σ) is a sphere of dimension $6g + 2b - 7$. In our case, $g = 1, b = 2$ so PML(Σ) is S^3.

With our coordinates, it is easy to exhibit this result by an explicit homeomorphism $\hat{V}/\mathbf{R}^+ \longrightarrow S^3$. As remarked above, we may regard $\hat{V}$ as

$$(\mathbf{R}^4 - \{(0,0,0,0)\})/ \approx$$

where $\approx$ is the equivalence relation

$$(q_1, p_1, q_2, p_2) \approx (-q_1, -p_1, q_2, p_2) \approx (q_1, p_1, -q_2, -p_2).$$

Since $\mathbf{R}^2/(q,p) \approx (-q,-p)$ is homeomorphic to $\mathbf{R}^2$ we see that $\hat{V}$ is homeomorphic to $\mathbf{R}^2 \times \mathbf{R}^2 - \{(0,0,0,0)\}$, which is just $\mathbf{R}^4 - \{(0,0,0,0)\}$. Furthermore, the identifications made preserve rays through the origin and thus commute with the projectivization. Thus $\hat{V}/\mathbf{R}^+$ is homeomorphic to $(\mathbf{R}^4 - \{(0,0,0,0)\})/\mathbf{R}^+$, which is S^3 as claimed.

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